Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS METHODS

Calculator-free

ATAR course examination 2023

Marking key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

Section One: Calculator-free **35% (53 Marks)**

Question 1 **(8 marks)**

(a) Consider the function $f(x) = x^3 e^{2x}$.

(i) Differentiate $f(x)$. (2 marks)

Solution
$f'(x) = \frac{d}{dx}(x^3)e^{2x} + x^3 \frac{d}{dx}(e^{2x})$ $= 3x^2 e^{2x} + 2x^3 e^{2x}$
Specific behaviours
✓ demonstrates use of the product rule ✓ obtains correct derivative

(ii) Determine the value of x for any stationary points of $f(x)$. (3 marks)

Solution
Setting $f'(x) = 0$ gives $0 = 3x^2 e^{2x} + 2x^3 e^{2x}$ $\Rightarrow 0 = x^2 e^{2x}(3 + 2x)$ $\Rightarrow x = 0, \ -\frac{3}{2}$
Specific behaviours
✓ sets $f'(x) = 0$ ✓ solves to obtain stationary point at $x = 0$ ✓ solves to obtain stationary point at $x = -\frac{3}{2}$

(b) Evaluate $\int_0^{\frac{\pi}{4}} \sin(2x+\pi)\,dx$. (3 marks)

Solution
$\int_0^{\frac{\pi}{4}} \sin(2x+\pi)\,dx = \left[-\frac{\cos(2x+\pi)}{2}\right]_0^{\frac{\pi}{4}}$ $= -\frac{\cos\left(\frac{3\pi}{2}\right)}{2} - \left(-\frac{\cos(\pi)}{2}\right)$ $= 0 - \left(\frac{1}{2}\right)$ $= -\frac{1}{2}$
Specific behaviours
✓ antidifferentiates correctly   ✓ correctly substitutes integration bounds   ✓ evaluates to obtain correct answer

Question 2 **(14 marks)**

Let $p = \ln(2)$, $q = \ln(3)$ and $r = \ln(5)$.

(a) Express each of the following in terms of p, q and/or r.

(i) $\ln(6)$ (2 marks)

Solution
$\ln(6) = \ln(2\times 3)$ $= \ln(2) + \ln(3)$ $= p + q$
Specific behaviours
✓ expresses $\ln(6)$ as the sum of $\ln(2)$ and $\ln(3)$ ✓ obtains correct expression in terms of p and q

(ii) $\ln(6.25)$ (3 marks)

Solution
$\ln(6.25) = \ln\left(\frac{25}{4}\right)$ $= \ln(25) - \ln(4)$ $= \ln\left(5^2\right) - \ln\left(2^2\right)$ $= 2\ln(5) - 2\ln(2)$ $= 2r - 2p$ Or $\ln(6.25) = \ln\left(\frac{25}{4}\right)$ $= 2\ln\left(\frac{5}{2}\right)$ $= 2(\ln(5) - \ln(2))$ $= 2(r - p)$
Specific behaviours
✓ expresses 6.25 as the fraction $\frac{25}{4}$ (or equivalent) ✓ applies log law to obtain a correct expression in terms of a difference of logs ✓ obtains correct expression in terms of p and r

(iii) $\int_2^3 \frac{d}{dx}\ln(x)\,dx$ (2 marks)

Solution
By the fundamental theorem of calculus $\int_2^3 \frac{d}{dx}\ln(x)\,dx = \left[\ln(x)\right]_2^3$ $= \ln(3) - \ln(2)$ $= q - p$
Specific behaviours
✓ evaluates the definite integral as $\ln(3) - \ln(2)$ ✓ correctly expresses answer in terms of p and q

(b) Evaluate e^{p+q}. (2 marks)

Solution
$e^{p+q} = e^p \times e^q$ $= e^{\ln(2)} \times e^{\ln(3)}$ $= 2 \times 3$ $= 6$
Specific behaviours
✓ correctly applies index law ✓ simplifies to obtain correct answer

(c) (i) Determine $\frac{d}{dx}\left(x\ln(x)\right)$. (1 mark)

Solution
$\frac{d}{dx}\left(x\ln(x)\right) = \ln(x) + x\frac{1}{x}$ $= \ln(x) + 1$
Specific behaviours
✓ differentiates correctly

Question 2 (continued)

(ii) Hence show that $\int \ln(x)\,dx = x\ln(x) - x + c$ where c is a constant. (2 marks)

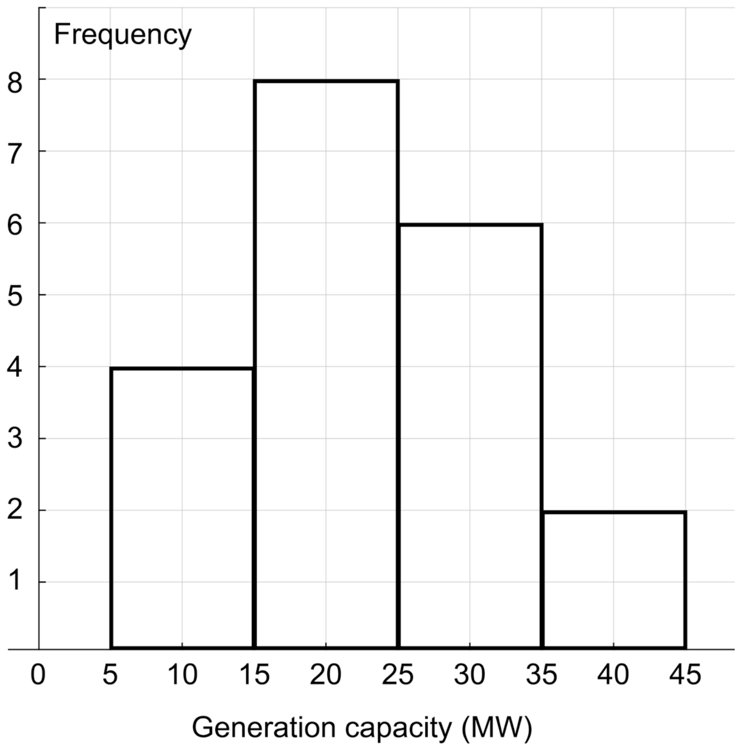
Solution
$\frac{d}{dx}\left(x\ln(x)\right) = \ln(x) + 1$ $\Rightarrow \int \frac{d}{dx}\left(x\ln(x)\right)dx = \int \ln(x)\,dx + \int 1\,dx$ $\Rightarrow x\ln(x) = \int \ln(x)\,dx + x + c$ $\Rightarrow \int \ln(x)\,dx = x\ln(x) - x + c$
Specific behaviours
✓ integrates both sides of the result from part (c)(i) and correctly evaluates $\int 1\,dx$ ✓ evaluates $\int \frac{d}{dx}\left(x\ln(x)\right)dx$ and applies valid mathematical operations to obtain required expression

(iii) Evaluate $\int_1^3 \ln(x)\,dx$ in terms of p, q and/or r. (2 marks)

Solution
$\int_1^3 \ln(x)\,dx = \left[x\ln(x) - x\right]_1^3$ $= 3\ln(3) - 3 - \left(\ln(1) - 1\right)$ $= 3q - 2$
Specific behaviours
✓ applies fundamental theorem of calculus to evaluate definite integral ✓ simplifies to obtain correct answer

Question 3 **(10 marks)**

Solcolwa is a green energy company that owns 20 solar farms across Western Australia. The generation capacities, in megawatts (MW), of the solar farms are displayed in the histogram below.



Suppose that one of the Solcolwa solar farms is selected at random. Let the random variable W denote the generation capacity of the randomly-selected solar farm.

(a) Complete the following table of cumulative probabilities for W. (2 marks)

Solution

w	5	15	25	35	45
$P(W \leq w)$	**0**	**0.2**	**0.6**	**0.9**	**1**

Specific behaviours

✓ correctly calculates at least three probabilities
✓ correctly calculates all probabilities

Question 3 (continued)

(b) Determine $P(W \geq 35)$. (1 mark)

Solution
$P(W \geq 35) = 1 - P(W \leq 35)$ $= 1 - 0.9$ $= 0.1$ Or $P(W \geq 35) = \frac{2}{20}$ $= 0.1$
Specific behaviours
✓ correctly calculates probability

(c) (i) estimate $P(W \geq 20)$. (2 marks)

Solution
Using the table of cumulative probabilities and linear interpolation: $P(W \geq 20) = 1 - P(W \leq 20)$ $= 1 - \frac{0.2 + 0.6}{2}$ $= 1 - 0.4$ $= 0.6$
Specific behaviours
✓ uses linear interpolation to estimate $P(W \leq 20)$ ✓ calculates correct probability

Alternate solution
Using the histogram and linear interpolation: $P(W \geq 20) = \frac{2 + 6 + \left(\frac{1}{2} \times 8\right)}{20}$ $= 0.6$
Specific behaviours
✓ determines number of solar farms with generating capacity between 20 and 25   ✓ calculates correct probability

(ii) estimate the expected value $E(W)$. (2 marks)

Solution
$E(W) = 10\times 0.2 + 20\times 0.4 + 30\times 0.3 + 40\times 0.1$ $= 2 + 8 + 9 + 4$ $= 23$
Specific behaviours
✓ writes correct expression for the expected value ✓ calculates correct expected value

(d) Given that W and Y have variances $Var(W) = 81$ and $Var(Y) = 324$, determine the expected value $E(Y)$. (3 marks)

Solution
Given that $Y = aW$ it follows that $Var(Y) = a^2 Var(W)$ $\Rightarrow 324 = 81a^2$ $\Rightarrow a^2 = 4$ $a = 2$ Hence $E(Y) = aE(W)$ $= 2\times 23$ $= 46$
Specific behaviours
✓ states correct relationship between $Var(W)$ and $Var(Y)$ ✓ calculates correct value of a ✓ calculates correct expected value

Question 4 **(8 marks)**

An internet search engine uses a logarithmic scale to rank the importance of internet websites. If a website has S visits each week, the site rank, R, is given by

$$R = 2\log_{10}\left(\frac{S}{S_0}\right)$$

where S_0 is the reference value (the same for all websites). The reference value is the minimum number of visits per week required for a website to register on the site rank scale.

(a) Determine the site rank for a website whose weekly visits are one hundred times the reference value. (2 marks)

Solution
$R = 2\log_{10}\left(\frac{100S_0}{S_0}\right)$ $= 2\log_{10}100$ $= 2\log_{10}10^2$ $= 4\log_{10}10$ $= 4$
Specific behaviours
✓ substitutes correctly to obtain $R = 2\log_{10}100$ ✓ simplifies to obtain correct answer

(b) Given that a site rank of 12 is assigned to a website with 1.5 billion (1.5×10^9) visits per week, determine the value of S_0. (3 marks)

Solution
Substituting into the equation above $12 = 2\log_{10}\left(\frac{1.5\times10^9}{S_0}\right)$ $\Rightarrow 6 = \log_{10}\left(\frac{1.5\times10^9}{S_0}\right)$ $\Rightarrow \frac{1.5\times10^9}{S_0} = 10^6$ $\Rightarrow S_0 = \frac{1.5\times10^9}{10^6}$ $= 1.5\times10^3$ $= 1500$
Specific behaviours
✓ correctly substitutes $R = 12$ and $S = 1.5 \times 10^9$ into the equation ✓ correctly rearranges the equation into the equivalent exponential expression ✓ solves for the correct value of S_0

(c) The plot of $y = \log_{10}(x)$ is shown below. If a website has a site rank of 3.2, use the plot and your answer from part (b) to approximate the website's number of weekly visits.

(3 marks)

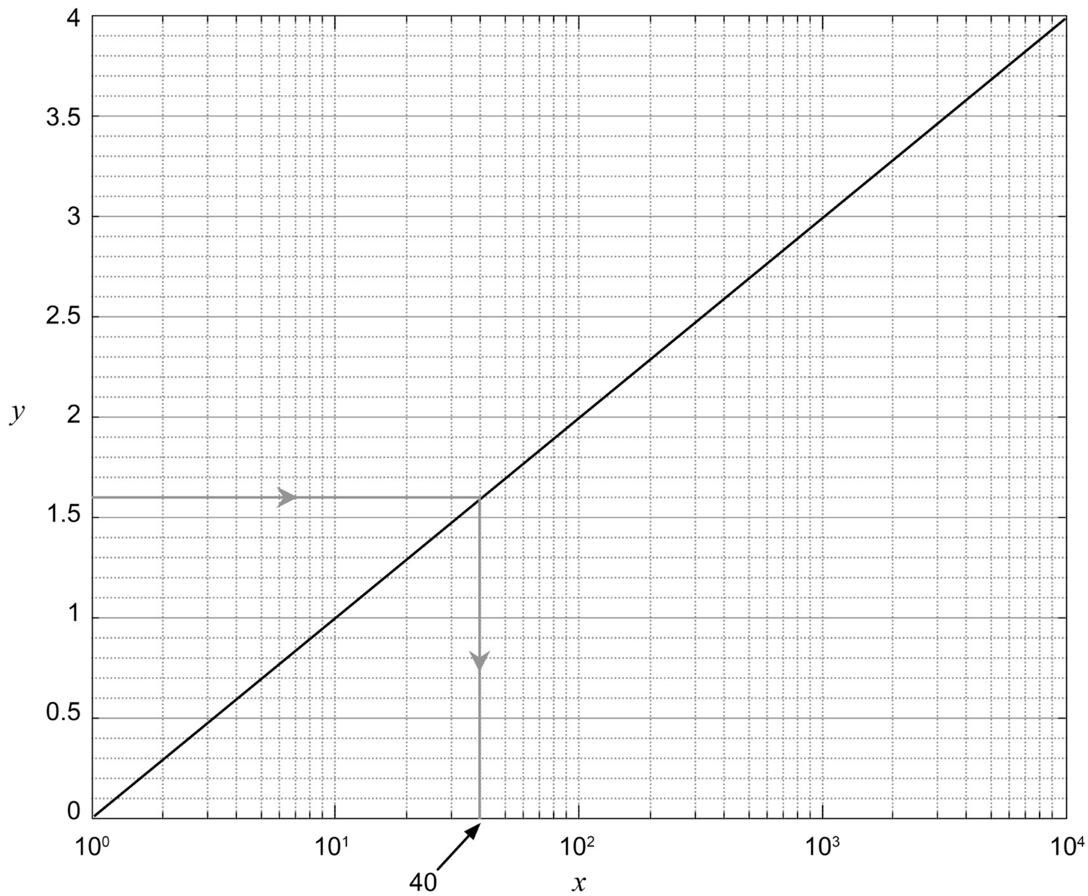
Solution

Substituting $R = 3.2$ into the site rank equation gives

$$3.2 = 2\log_{10}\left(\frac{S}{1500}\right)$$

$$\Rightarrow 1.6 = \log_{10}\left(\frac{S}{1500}\right)$$

From the graph, when $y = 1.6,\ x \approx 40$.



Hence

$$\frac{S}{1500} \approx 40$$

$$\Rightarrow S \approx 1500 \times 40$$

$$= 60\ 000$$

so the number of weekly visits is approximate 60 000.

Specific behaviours

✓ identifies the need to solve $1.6 = \log_{10}(x)$

✓ uses the graph to determine that when $y = 1.6,\ x \approx 40$

✓ determines the correct number of weekly visits

Question 5 **(13 marks)**

The table below contains values of the polynomial function $f(x)$, its first and second derivatives, and the function $F(x)=\int_0^x f(t)dt$ for $x=0,1,2,3,4,5,6$.

$f(x)$ has no stationary points at non-integer values of x, and the letters a,b,c,d and e represent unspecified constants.

(a) Evaluate $\frac{d}{dx}\left(f(x)^2\right)$ when $x=2$. (2 marks)

Solution
By the chain rule $\frac{d}{dx}\left(f(x)^2\right)=2f(x)f'(x)$ Substituting $x=2$ gives $\left.\frac{d}{dx}\left(f(x)^2\right)\right\|_{x=2}=2f(2)f'(2)$ $=2\times4\times-4$ $=-32$ Or By the product rule $\frac{d}{dx}\left(f(x)^2\right)=\frac{d}{dx}\left(f(x)f(x)\right)$ $=f(x)f'(x)+f(x)f'(x)$ Substituting $x=2$ gives $\left.\frac{d}{dx}\left(f(x)^2\right)\right\|_{x=2}=f(2)f'(2)+f(2)f'(2)$ $=4\times-4+4\times-4$ $=-16-16$ $=-32$
Specific behaviours
✓ correctly applies the chain rule or product rule ✓ calculates correct derivative

(b) Evaluate $\int_2^4 \left(f(x)+2 \right) dx$. (3 marks)

Solution
$\int_2^4 \left(f(x)+2 \right) dx = \int_2^4 f(x)\, dx + \int_2^4 2\, dx$ $= F(4)-F(2)+\left[2x\right]_2^4$ $=12.8-10.4+(8-4)$ $=6.4$
Specific behaviours
✓ correctly applies linearity of definite integrals   ✓ correctly applies fundamental theorem to first integral   ✓ correctly evaluates $\int_2^4 2\, dx$ and obtains correct answer

(c) Evaluate $\frac{d}{dx}\int_2^x f(t)\, dt$ when $x=2$. (2 marks)

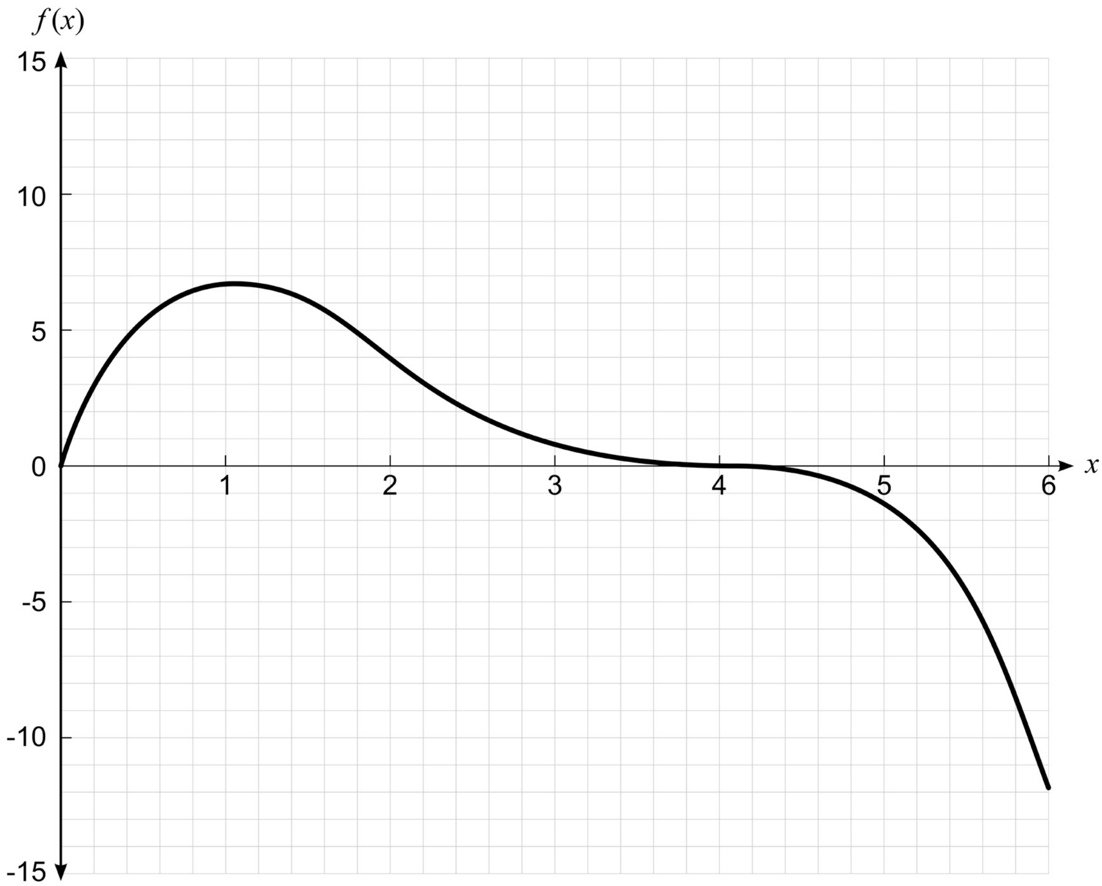
Solution
By the fundamental theorem of calculus $\frac{d}{dx}\int_2^x f(t)\, dt = f(x)$ Substituting $x=2$ gives $\left.\frac{d}{dx}\int_2^x f(t)\, dt\right\|_{x=2} = f(2)$ $=4$
Specific behaviours
✓ correctly applies the fundamental theorem of calculus   ✓ correctly evaluates for $x=2$

(d) Determine the x-coordinate of any stationary points and whether they are local maxima, local minima or inflection points. Justify your answer. (3 marks)

Solution
Stationary points are when $f'(x)=0$, hence $x=1$ and $x=4$ are the stationary points.   Since $f''(1)=-9$ it follows that $x=1$ is a local maximum by the second derivative test.   Since $f''(4)=0$ the second derivative test fails. Since the gradient of f is negative on both sides of $x=4$ (i.e. $f'(3)=-2<0$, $f'(5)=-4<0$, and there are no stationary points for non-integer values of x) it follows that $x=4$ is a horizontal point of inflection.
Specific behaviours
✓ correctly identifies the coordinates $x=1$ and $x=4$ as stationary points   ✓ concludes that $x=1$ is a local maximum with correct justification   ✓ concludes that $x=4$ is an inflection point with correct justification

Question 5 (continued)

(e) Sketch a possible graph of $f(x)$ for $0 \le x \le 6$ on the axes below. (3 marks)

Solution

Specific behaviours
✓ graph passes through the points $(2, 4)$ and $(4, 0)$ ✓ graph includes local maximum at $x = 1$ and a horizontal point of inflection at $x = 4$ ✓ concavity of graph is correct including inflection point at $x = 2$

Published by the School Curriculum and Standards Authority of Western Australia
303 Sevenoaks Street
CANNINGTON WA 6107