Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS METHODS

Calculator-free

ATAR course examination 2021

Marking key

Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

2021/67365

Section One: Calculator-free **35% (51 Marks)**

Question 1 **(9 marks)**

(a) Differentiate $\dfrac{3x+1}{x^3}$ and simplify your answer. (3 marks)

Solution
$\dfrac{d}{dx}\left(\dfrac{3x+1}{x^3}\right)=\dfrac{x^3(3)-3x^2(3x+1)}{x^6}$ $=\dfrac{3x^3-9x^3-3x^2}{x^6}$ $=\dfrac{-6x-3}{x^4}$
Specific behaviours
✓ recognises the need for the quotient rule   ✓ correctly differentiate the expression   ✓ simplifies the result

(b) Let $f'(x)=x\ln(2x)$. Determine a simplified expression for the rate of change of $f'(x)$. (3 marks)

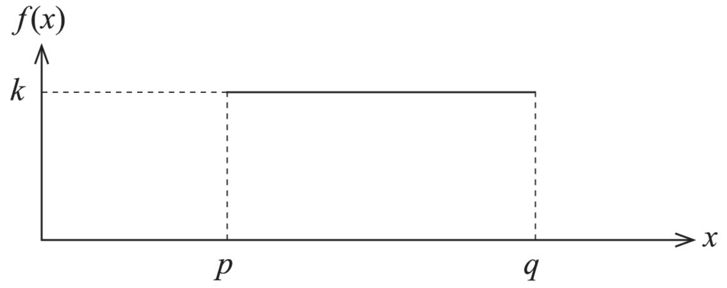
Solution
$f''(x)=x\times\dfrac{2}{2x}+1\times\ln(2x)$ $=1+\ln(2x)$
Specific behaviours
✓ identifies the rate of change as $f''(x)$ ✓ correctly determines $f''(x)$ ✓ simplifies the expression for $f''(x)$

(c) Given that $g'(x)=4e^{2x}$ and $g(1)=0$, determine $g(5)$. (3 marks)

Solution
$g(x)=2e^{2x}+c$ Since $g(1)=0$, $0=2e^2+c$ $\therefore c=-2e^2$ $\therefore g(x)=2e^{2x}-2e^2$ $\therefore g(5)=2e^{10}-2e^2$
Specific behaviours
✓ states an expression for $g(x)$, including the constant of integration   ✓ correctly determines the constant   ✓ correctly determines $g(5)$ as the final solution

Question 2 **(10 marks)**

It takes Nahyun between 15 and 40 minutes to get to school each day, depending on traffic conditions. Nahyun leaves home for school at 8.00 am each school day. Let the random variable X be the time, in minutes after 8:00 am, that Nahyun arrives at school. The probability density function of X is shown below.



(a) What is the name of this type of distribution? (1 mark)

Solution
Continuous uniform distribution
Specific behaviours
✓ correctly states the name of the distribution

(b) Determine:

(i) the values of p, q and k (2 marks)

Solution
$p = 15$ $q = 40$ $k = \frac{1}{25}$
Specific behaviours
✓ correctly states the values of p and q ✓ correctly states the value of k

(ii) the expected value of X (1 mark)

Solution
$E(x) = \frac{40+15}{2}$ $= 27.5$ minutes
Specific behaviours
✓ correctly states the expected value

Question 2 (continued)

(iii) the probability that Nahyun arrives at school before 8:25 am. (2 marks)

Solution
$\mathrm{P}(X < 25) = \dfrac{25-15}{25}$ $= \dfrac{10}{25}\left\{= \dfrac{2}{5}\right\}$
Specific behaviours
✓ identifies the area between 15 and 25 is required   ✓ calculates the correct probability (simplified probability not required)

Nahyun will be late for her first class if she arrives at school after 8:28 am. Otherwise, she will not be late.

(c) If Nahyun is not late for her first class, what is the probability that she arrives after 8:25 am? (2 marks)

Solution
$P(X > 25 \mid X < 28) = \dfrac{3}{13}$
Specific behaviours
✓ correctly identifies the situation is a conditional probability   ✓ determines the correct probability

(d) If Nahyun only wants to be late for her first class at most 4% of the time, what time should she leave home, assuming the 15 to 40 minute travel time remains the same? (2 marks)

Solution
$4\% = \dfrac{4}{100} = \dfrac{1}{25}$ $\therefore$ leaves 39 minutes before 8:28 am   She should leave home at 7:49 am
Specific behaviours
✓ determines 4% = a probability of $\dfrac{1}{25}$ ✓ correctly determines the time

Question 3 **(3 marks)**

Given that $\ln(2) \approx 0.693$, use the increments formula to determine an approximation for $\ln(2.02)$.

Solution
Let $y = \ln(x)$ Then $x = 2, \delta x = 0.02$ $\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$ $\therefore \delta y \approx \frac{dy}{dx} \times \delta x$ $= \frac{1}{x} \times \delta x$ $= \frac{0.02}{2}$ $= 0.01$ $\therefore \ln(2.02) \approx \ln(2) + 0.01$ $= 0.703$
Specific behaviours
✓ correctly determines δx ✓ correctly determines δy ✓ determines correct approximation

Question 4 **(7 marks)**

Determine the following:

(a) $\int\left(2x^2 - x^3\right)dx$ (2 marks)

Solution
$\int\left(2x^2 - x^3\right)dx = \frac{2x^3}{3} - \frac{x^4}{4} + c$
Specific behaviours
✓ integrates correctly ✓ includes the constant of integration

(b) $\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{3-\cos(x)}\,dx$ (3 marks)

Solution
$\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{3-\cos(x)}\,dx = \left[\ln(3-\cos(x))\right]_0^{\frac{\pi}{2}}$ $= \ln(3-\cos(\frac{\pi}{2})) - \ln(3-\cos(0))$ $= \ln 3 - \ln 2$ $= \ln\left(\frac{3}{2}\right)$
Specific behaviours
✓ correctly integrates ✓ substitutes limits ✓ determines the correct simplified answer

(c) $\frac{d}{dy}\int_{-1}^{y} 3x^2\cos(2x)\,dx$ (2 marks)

Solution
$\frac{d}{dy}\int_{-1}^{y} 3x^2\cos(2x)\,dx = 3y^2\cos(2y)$
Specific behaviours
✓ identifies the need for the Fundamental Theorem of Calculus ✓ states the correct result

Question 5 **(6 marks)**

(a) Determine the area between the parabola $y = x^2 - x + 3$ and the straight line $y = x + 3$. (4 marks)

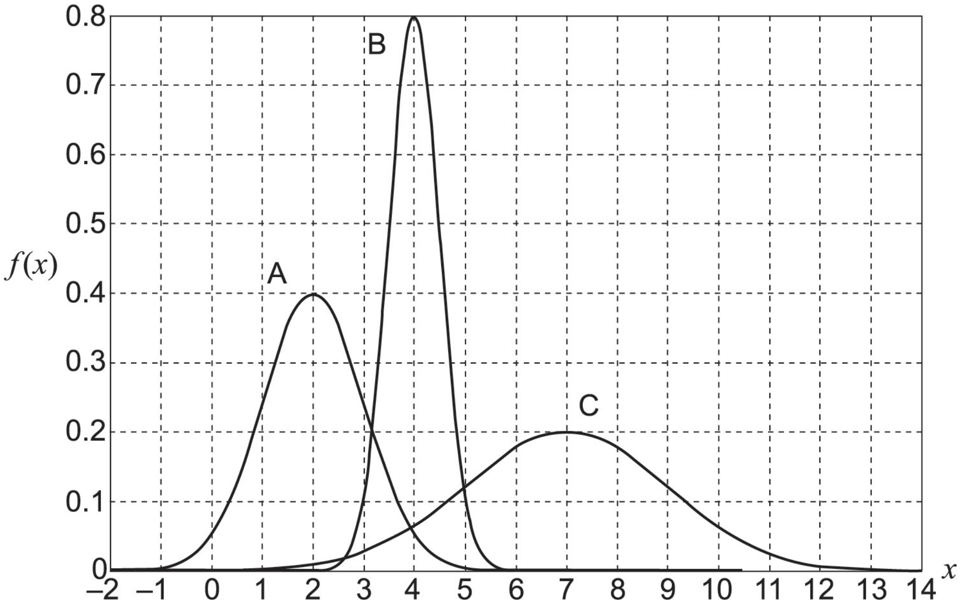
Solution
Point of intersection: $x^2 - x + 3 = x + 3$ $x^2 - 2x = 0$ $x(x-2) = 0$ $\therefore x = 0, 2$ $\int_0^2 \left[(x+3) - (x^2 - x + 3)\right] dx$ $= \int_0^2 \left[2x - x^2\right] dx$ $= \left[x^2 - \frac{x^3}{3}\right]_0^2$ $= 4 - \frac{8}{3}$ $= \frac{4}{3}\ \text{units}^2$
Specific behaviours
✓ determines x coordinates of the points of intersection   ✓ states correct integral for area   ✓ evaluates integral   ✓ determines correct area

(b) The area between the parabola $y = x^2 - x - 2$ and the straight line $y = x - 2$ is the same as the area determined in part (a). Explain why this is the case. (2 marks)

Solution
Both graphs from part (a) have been vertically translated down 5 units.   The shape of both graphs is unchanged.   Therefore, the area between them remains unchanged.
Specific behaviours
✓ states both graphs have been translated in the same direction by the same amount   ✓ states both graphs retain the same shape

Question 6 **(7 marks)**

(a) The graphs of three normal distributions are displayed below. The distributions have been labelled A, B and C.



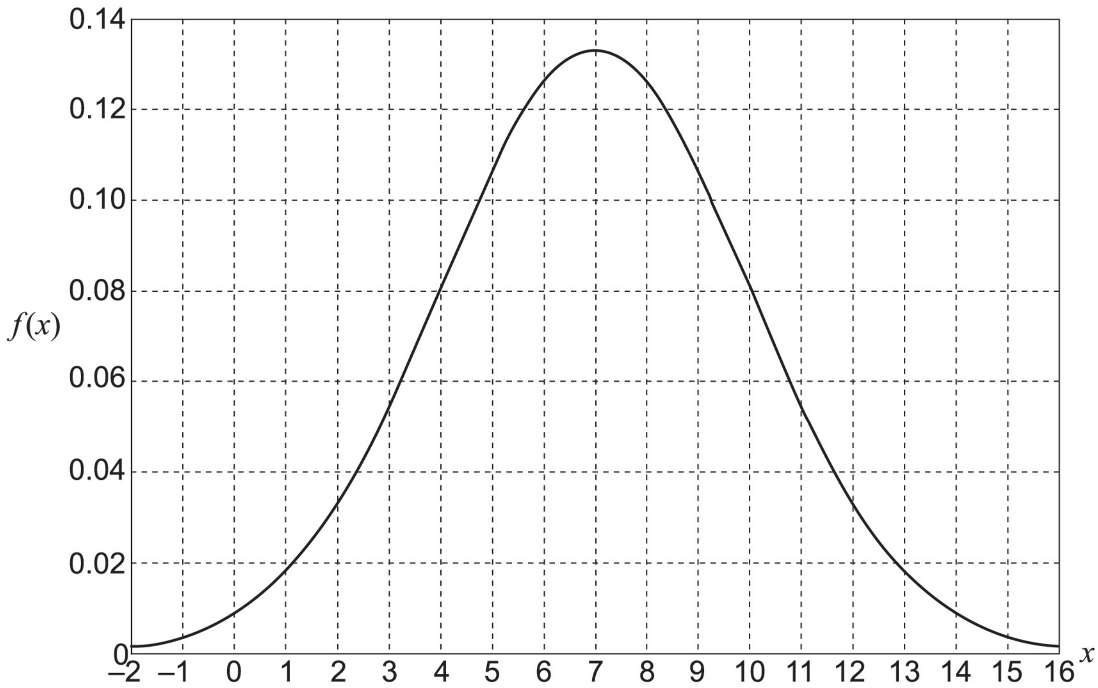
(i) What is the mean of distribution A? (1 mark)

Solution
Mean = 2
Specific behaviours
✓ determines mean of A

(ii) Which of the distributions has the largest standard deviation? Justify your answer. (1 mark)

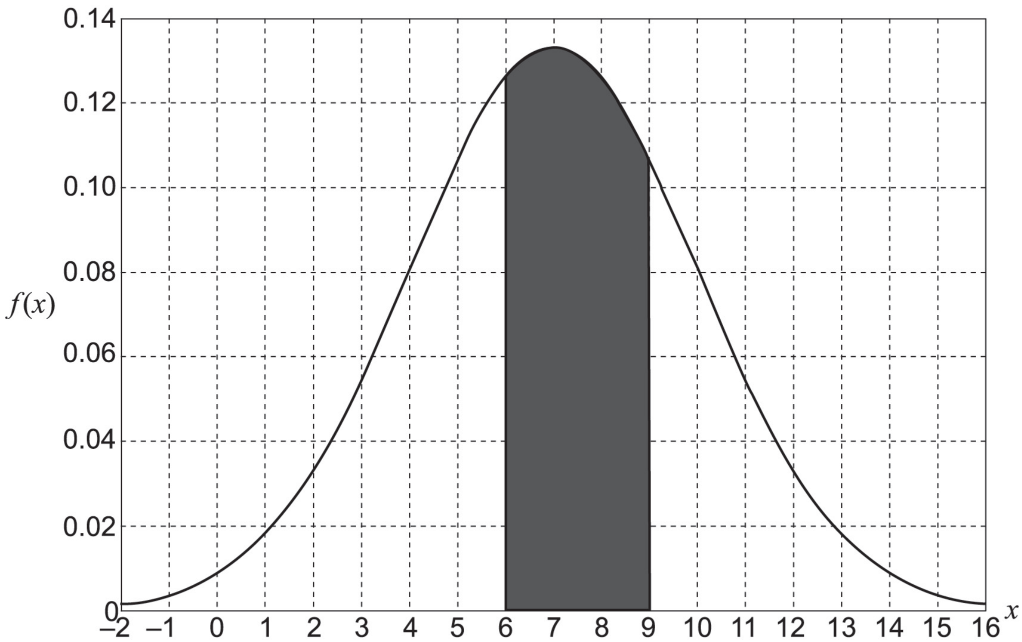
Solution
C has the largest standard deviation as it is the widest distribution.
Specific behaviours
✓ states that C has the largest standard deviation and provides correct justification

(b) A random variable X is normally distributed. The distribution of X is graphed below.



(i) Shade the region with area corresponding to $\mathrm{P}(6 \leq X \leq 9)$. (1 mark)

Solution



Specific behaviours

✓ shades correct region

Question 6 (continued)

(ii) Is $P(6 \le X \le 9) \ge 0.5$? Justify your answer. (2 marks)

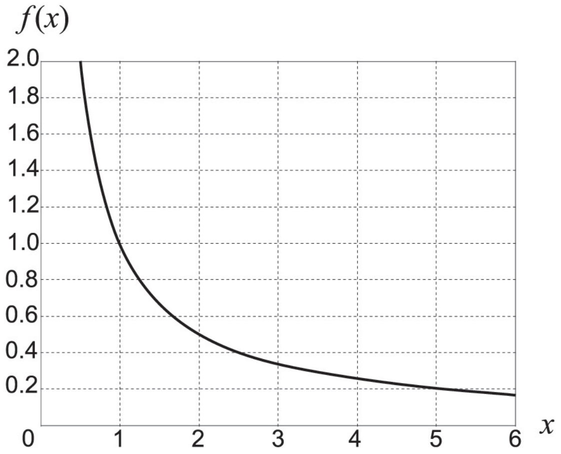
Solution
No. The total area below the probability density function is 1, and the region shaded above is less than half of that area (i.e. area is less than 0.5). Hence, it corresponds to a probability that is less than 0.5.
Specific behaviours
✓ states that the probability is not greater than or equal to 0.5 ✓ provides correct justification

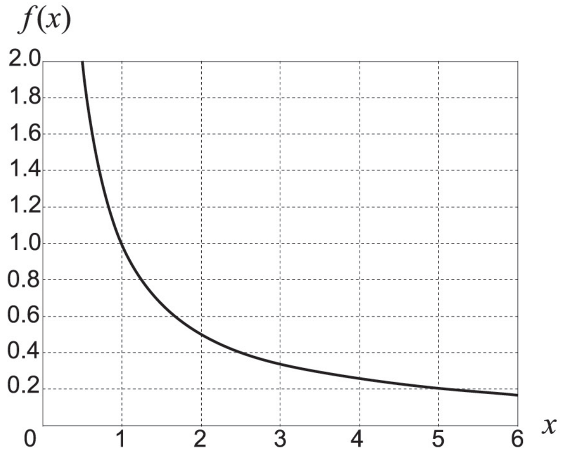
(c) A random variable Y has probability $P(Y \ge 2) > P(Y > 2)$. Explain whether it is possible for the distribution of Y to be normal or binomial. (2 marks)

Solution
Not normal: a continuous random variable has $P(Y \ge 2) = P(Y > 2)$. Since a normally distributed random variable is continuous it follows that Y is not a normally distributed random variable. Could be binomial: $P(Y \ge 2) > P(Y > 2)$ for a discrete random variable. Since the binomial distribution is discrete it follows that Y could be a binomially distributed random variable.
Specific behaviours
✓ states that Y could not be normal and provides a correct explanation ✓ states that Y could be binomial and provides a correct explanation

Question 7 **(9 marks)**

(a) Consider the function $f(x) = \frac{1}{x}$, graphed twice below.





(i) Shade **two** different regions (one on each graph above) each with area exactly $\ln(2)$. (2 marks)

Solution
Two distinct cases in which the upper bound is twice the lower bound. I would expect most to shade under the curve from $x = 1$ to $x = 2$, and then from $x = 2$ to $x = 4$. Other possibilities would be $x = 1.5$ to $x = 3$, $x = 2.5$ to $x = 5$ or $x = 3$ to $x = 6$.
Specific behaviours
✓ shades a region under the curve corresponding to $\ln(2)$ ✓ shades a second distinct region under the curve corresponding to $\ln(2)$

(ii) Given that

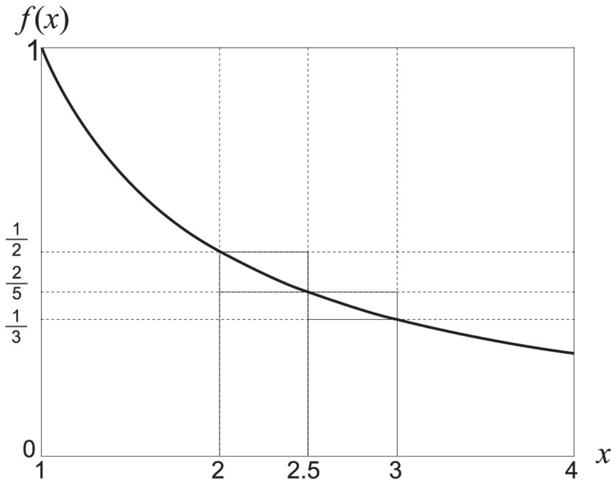
$$\int_a^b \frac{1}{x} dx = \ln(3)$$

what is the relationship between a and b? (2 marks)

Solution
$Area = \int_a^b \frac{1}{x} dx = \ln\left(\frac{b}{a}\right)$ $= \ln 3$ So, $b = 3a$.
Specific behaviours
✓ obtains the correct integral in terms of a and b ✓ states the relationship between a and b

Question 7 (continued)

(b) Another graph of $f(x) = \dfrac{1}{x}$ is shown below.



(i) By considering the areas of the rectangles shown, demonstrate and explain why $\dfrac{11}{30} < \displaystyle\int_2^3 \frac{1}{x}\,dx < \frac{9}{20}$. (3 marks)

Solution
Using the rectangles that estimate $y = \frac{1}{x}$ on the left side of each interval gives $$\int_2^3 \frac{1}{x}dx < \frac{1}{2}\times\frac{1}{2}+\frac{2}{5}\times\frac{1}{2}=\frac{9}{20}$$ This is an overestimate of the integral as the top of the rectangles lie above the graph. Using the rectangles that estimate $y = \frac{1}{x}$ on the right side of each interval gives $$\int_2^3 \frac{1}{x}dx > \frac{2}{5}\times\frac{1}{2}+\frac{1}{3}\times\frac{1}{2}=\frac{11}{30}$$ This is an underestimate of the integral as the top of the rectangles lie below the graph. Hence, $$\frac{11}{30} < \int_2^3 \frac{1}{x}dx < \frac{9}{20}$$
Specific behaviours
✓ approximates the integral using $\frac{1}{2}\times\frac{1}{2}+\frac{2}{5}\times\frac{1}{2}=\frac{9}{20}$ ✓ approximates the integral using $\frac{2}{5}\times\frac{1}{2}+\frac{1}{3}\times\frac{1}{2}=\frac{11}{30}$ ✓ explains why the first approximation is an overestimate and the second is an underestimate

(ii) Hence show that $\frac{11}{30} < \ln(1.5) < \frac{9}{20}$. (2 marks)

Solution
$\int_2^3 \frac{1}{x}\,dx = [\ln(x)]_2^3 = \ln(3) - \ln(2) = \ln(1.5)$ Hence $\frac{11}{30} < \ln(1.5) < \frac{9}{20}$
Specific behaviours
✓ correctly integrates $\frac{1}{x}$ and substitutes bounds to obtain $\ln(3) - \ln(2)$ ✓ applies log law to obtain $\ln(3) - \ln(2) = \ln(1.5)$

Published by the School Curriculum and Standards Authority of Western Australia
303 Sevenoaks Street
CANNINGTON WA 6107