Government of **Western Australia**
School Curriculum and Standards Authority

MATHEMATICS METHODS

Calculator-free

ATAR course examination 2020

Marking key

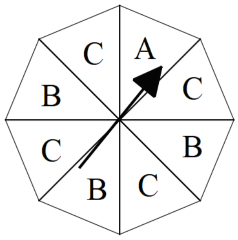
Marking keys are an explicit statement about what the examining panel expect of candidates when they respond to particular examination items. They help ensure a consistent interpretation of the criteria that guide the awarding of marks.

2020/66937

Section One: Calculator-free **35% (51 Marks)**

Question 1 **(6 marks)**

Ashley and Xavier are playing a board game that requires them to use the spinner shown below.



The player spins the arrowhead and the result is where the arrowhead is pointing when it stops moving. The above diagram is showing a result of A.

(a) If the spinner is spun three times, what is the probability that B is never a result? (1 mark)

Solution
$P(\text{not B}) = \left(\frac{5}{8}\right)^3$ $= \frac{125}{512}$
Specific behaviours
✓ determines the correct probability

Let the random variable X be defined as the number of times B is the result when the spinner is spun three times.

(b) Complete the table below showing the probability distribution of X. (3 marks)

Solution

$X \sim Bin\left(3, \frac{3}{8}\right)$

x	0	1	2	3
$P(X = x)$	$\left(\frac{5}{8}\right)^3$ or $\frac{125}{512}$	$\binom{3}{1}\left(\frac{5}{8}\right)^2\left(\frac{3}{8}\right)$ or $\frac{225}{512}$	$\binom{3}{2}\left(\frac{5}{8}\right)\left(\frac{3}{8}\right)^2$ or $\frac{135}{512}$	$\left(\frac{3}{8}\right)^3$ or $\frac{27}{512}$

Specific behaviours

✓ recognises the distribution of X as binomial
✓ determines the correct probability for $x = 1$, 2 or 3
✓ determines the correct probability for remaining entries

(c) Determine the mean and variance of the above distribution. (2 marks)

Solution	
$\text{mean} = np$ $= 3 \times \frac{3}{8}$ $= \frac{9}{8} \quad \{1.125\}$	$\text{variance} = np(1-p)$ $= \frac{9}{8} \times \frac{5}{8}$ $= \frac{45}{64}$
Specific behaviours	
✓ determines the mean ✓ determines the variance	

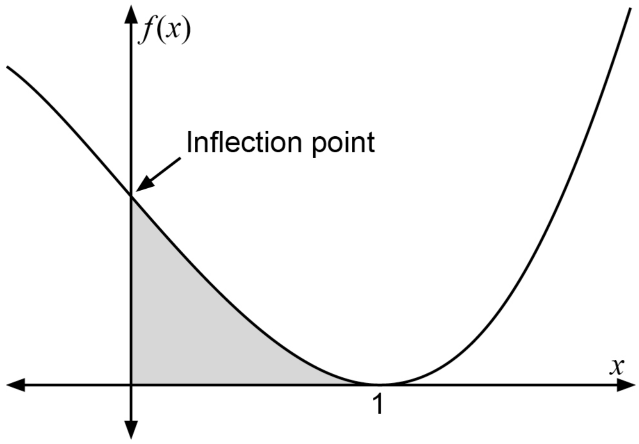
Question 2 **(4 marks)**

If $h(x) = \dfrac{e^{-x}}{\cos x}$, then evaluate $h'(\pi)$.

Solution
$$h'(x) = \frac{-e^{-x}(\cos x) - e^{-x} \times (-\sin x)}{(\cos x)^2}$$ $$h'(\pi) = \frac{-e^{-\pi}(\cos \pi) - e^{-\pi} \times (-\sin \pi)}{(\cos \pi)^2}$$ $$= \frac{-e^{-\pi} \times -1 - e^{-\pi} \times 0}{(-1)^2}$$ $$= e^{-\pi}$$
Specific behaviours
✓ demonstrates use of the quotient rule ✓ differentiates $\cos x$ and e^{-x} correctly ✓ substitutes $x = \pi$ correctly ✓ evaluates correctly

Question 3 **(7 marks)**

The graph of the cubic function $f(x) = ax^3 + bx^2 + cx + d$ is shown below. A turning point is located at (1, 0) and the shaded region shown on the graph has an area of $\frac{3}{2}$ units^2.

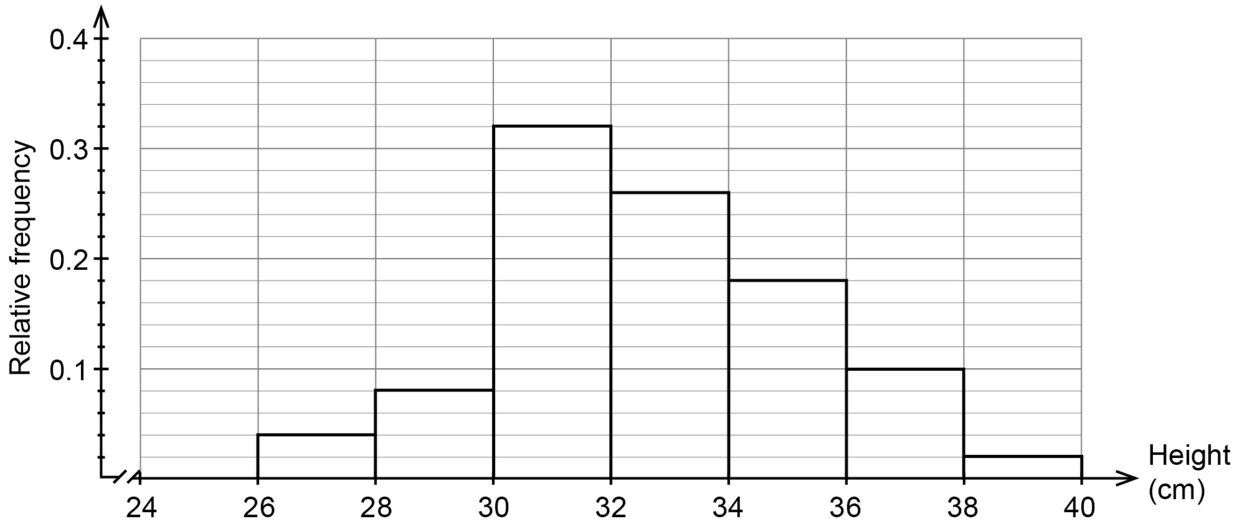


Use the above information to determine the values of a, b, c and d.

Solution
Firstly, note that $$f'(x) = 3ax^2 + 2bx + c$$ and $$f''(x) = 6ax + 2b$$ Given that there is an inflection point at $x = 0$ it follows that $f''(0) = 0$. Hence $$0 = 2b$$ $$b = 0$$ Given that there is a turning point at $x = 1$ it follows that $f'(1) = 0$. Hence $$0 = 3a + c$$ $$c = -3a$$ Given that there is an x-intercept at $x = 1$ it follows that $f(1) = 0$. Hence $$0 = a - 3a + d$$ $$d = 2a$$ Finally, given that the area of the shaded region is $\frac{3}{2}$ it follows that $\int_0^1 f(x)dx = \frac{3}{2}$. $$a\int_0^1 x^3 - 3x + 2\ dx = \frac{3}{2}$$ $$a\left[\frac{x^4}{4} - \frac{3x^2}{2} + 2x\right]_0^1 = \frac{3}{2}$$ $$\frac{3a}{4} = \frac{3}{2}$$ $$a = 2$$ Hence $a = 2$, $b = 0$, $c = -6$ and $d = 4$.
Specific behaviours
✓ states the first and second derivatives of f ✓ recognises that $f''(0) = 0$ and hence that $b = 0$ ✓ recognises that $f'(1) = 0$ and hence that $c = -3a$ ✓ recognises that $f(1) = 0$ and hence that $d = 2a$ (or $d = -a - c$) ✓ recognises that $\int_0^1 f(x)dx = \frac{3}{2}$ ✓ evaluates definite integral to determine that $a = 2$ ✓ solves for the values of c and d

Question 4 **(9 marks)**

The heights reached by a species of small plant at maturity are measured by a team of biologists. The results are shown in the histogram of relative frequencies below.



(a) Determine the probability that a mature plant of this species reaches no higher than 30 cm. (1 mark)

Solution
$P(h \le 30) = 0.04 + 0.08$ $= 0.12$
Specific behaviours
✓ determines the correct probability

(b) If a mature plant reaches a height of at least 32 cm, what is the probability that its height reaches above 38 cm? (2 marks)

Solution
$P(h \ge 38 \mid h \ge 32) = \dfrac{0.02}{0.56} = \dfrac{1}{28}$
Specific behaviours
✓ recognises conditional probability and determines the correct denominator of the conditional probability ✓ determines the correct probability as a fraction

Question 4 (continued)

Another team of biologists is studying the mature heights of a species of hedge. The height, h metres, has a probability density function, $d(h)$, as given below.

$$d(h)=\begin{cases}\dfrac{h-1}{5} & \text{for} \quad 1\le h\le 2\\ kh^2 & \text{for} \quad 2<h\le 4\\ 0 & \text{otherwise}\end{cases}$$

(c) What percentage of hedges from this study reaches a mature height less than 2 m? (3 marks)

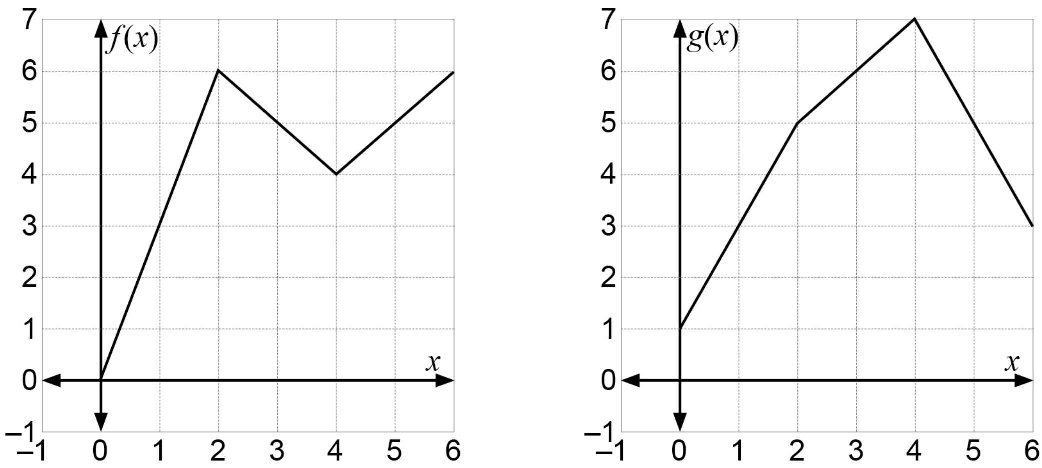
Solution
Probability of $1<h\le 2$: $\int_1^2 \frac{h-1}{5}\,dh=\frac{1}{5}\left[\frac{h^2}{2}-h\right]_1^2$ $=\frac{1}{5}\left[2-2-\frac{1}{2}+1\right]$ $=\frac{1}{10}$ 10% reach a height of less than 2 m
Specific behaviours
✓ recognises the need to integrate the first equation of the PDF from 1 to 2   ✓ antidifferentiates the first equation correctly   ✓ determines the correct percentage

(d) Determine the value of k. (3 marks)

Solution
Probability of $2<h\le 4=1-\frac{1}{10}=\frac{9}{10}$ $\therefore \frac{9}{10}=\int_2^4 kh^2\,dh$ $\frac{9}{10}=k\left[\frac{h^3}{3}\right]_2^4$ $\frac{9}{10}=k\left[\frac{64-8}{3}\right]$ $k=\frac{27}{560}$
Specific behaviours
✓ recognises the need to integrate the second equation of the PDF from 2 to 4 and equates to the complement of part (c), $\frac{9}{10}$ ✓ antidifferentiates the second equation correctly   ✓ determines the value of k

Question 5 **(5 marks)**

The graphs of the functions f and g are displayed below.



(a) Evaluate the derivative of $f(x)$ at $x = 3$. (1 mark)

Solution
$f'(3) = -1$
Specific behaviours
✓ states correct derivative

(b) Evaluate the derivative of $f(x)\,g(x)$ at $x = 5$. (2 marks)

Solution
$(fg)'(5) = f'(5)g(5) + g'(5)f(5)$ $= (1)(5) + (-2)(5)$ $= -5$
Specific behaviours
✓ uses product rule to express derivative ✓ states correct derivative

(c) Evaluate the derivative of $f\left(g(x)\right)$ at $x = 1$. (2 marks)

Solution
$f\left(g(x)\right)'\big\|_1 = f'\left(g(1)\right)g'(1)$ $= f'(3)2$ $= (-1)2$ $= -2$
Specific behaviours
✓ uses chain rule to express derivative ✓ states correct derivative

Question 6 **(7 marks)**

Consider the function $f(x) = \ln(x)$. The function $g(x) = f(x) + a$ is a vertical translation of f by a units.

(a) Express the function $g(x) = \ln(4x)$ in terms of a vertical translation of f (i.e. in the form $g(x) = f(x) + a$), stating the number of units that f is translated. (2 marks)

Solution
$g(x) = \ln(4x)$ $= \ln(4) + \ln(x)$ $= f(x) + \ln(4)$ f is translated vertically (upward) by $\ln(4)$ units.
Specific behaviours
✓ expresses $g(x)$ as a sum of logs ✓ recognises a vertical translation by $\ln(4)$ units

The function $h(x) = c f(x)$ is a vertical dilation of f by a scale factor of c.

(b) Express the function $h(x) = \ln(\sqrt{x})$ in terms of a vertical dilation of f, stating the scale factor. (2 marks)

Solution
$h(x) = \ln(\sqrt{\text{x}})$ $= \ln(x^{0.5})$ $= 0.5\ln(x)$ $= 0.5f(x)$ f is scaled vertically by a factor of 0.5.
Specific behaviours
✓ expresses h as a product involving $\ln(x)$ ✓ recognises a vertical scaling by a scale factor of 0.5

The function $p(x) = f(bx)$ is a horizontal dilation of f by a scale factor of $\frac{1}{b}$.

(c) Express the function $p(x) = \ln(x) + 4$ in terms of a horizontal dilation of f, stating the scale factor. (3 marks)

Solution
$p(x) = \ln(x) + 4$ $= \ln(x) + 4\ln(e)$ $= \ln(x) + \ln(e^4)$ $= \ln(e^4 x)$ $= f(e^4 x)$ f is scaled horizontally by a scale factor of e^{-4}.
Specific behaviours
✓ expresses 4 as $4\ln(e)$ ✓ expresses p using a single logarithm ✓ states horizontal scale factor

Question 7 **(13 marks)**

Consider the function $f(x) = e^{2x} - 4e^{x}$.

(a) Determine the coordinates of the x-intercept(s) of f. You may wish to consider the factorised version of $f: f(x) = e^{x}(e^{x} - 4)$. (3 marks)

Solution
Solve $f(x) = 0$ $0 = e^{x}(e^{x} - 4)$ $e^{x} = 4$ $x = \ln(4)$ Hence x-intercept at $(\ln(4), 0)$.
Specific behaviours
✓ states correct equation to be solved ✓ solves for x ✓ states coordinates

(b) Show that there is only one turning point on the graph of f, which is located at $(\ln(2), -4)$. (3 marks)

Solution
$f'(x) = 2e^{2x} - 4e^{x}$ Solve $f'(x) = 0$ $0 = 2e^{2x} - 4e^{x}$ $= 2e^{x}(e^{x} - 2)$ $e^{x} = 2$ $x = \ln(2)$ Substitute $x = \ln(2)$ into $f(x)$ $f(\ln(2)) = e^{2\ln(2)} - 4e^{\ln(2)}$ $= e^{\ln(4)} - 4e^{\ln(2)}$ $= 4 - 8$ $= -4$ Turning point at $(\ln(2), -4)$.
Specific behaviours
✓ differentiates $f(x)$ correctly and equates to 0 ✓ shows the steps required to solve for x ✓ demonstrates the use of log laws to determine the y-coordinate

Question 7 (continued)

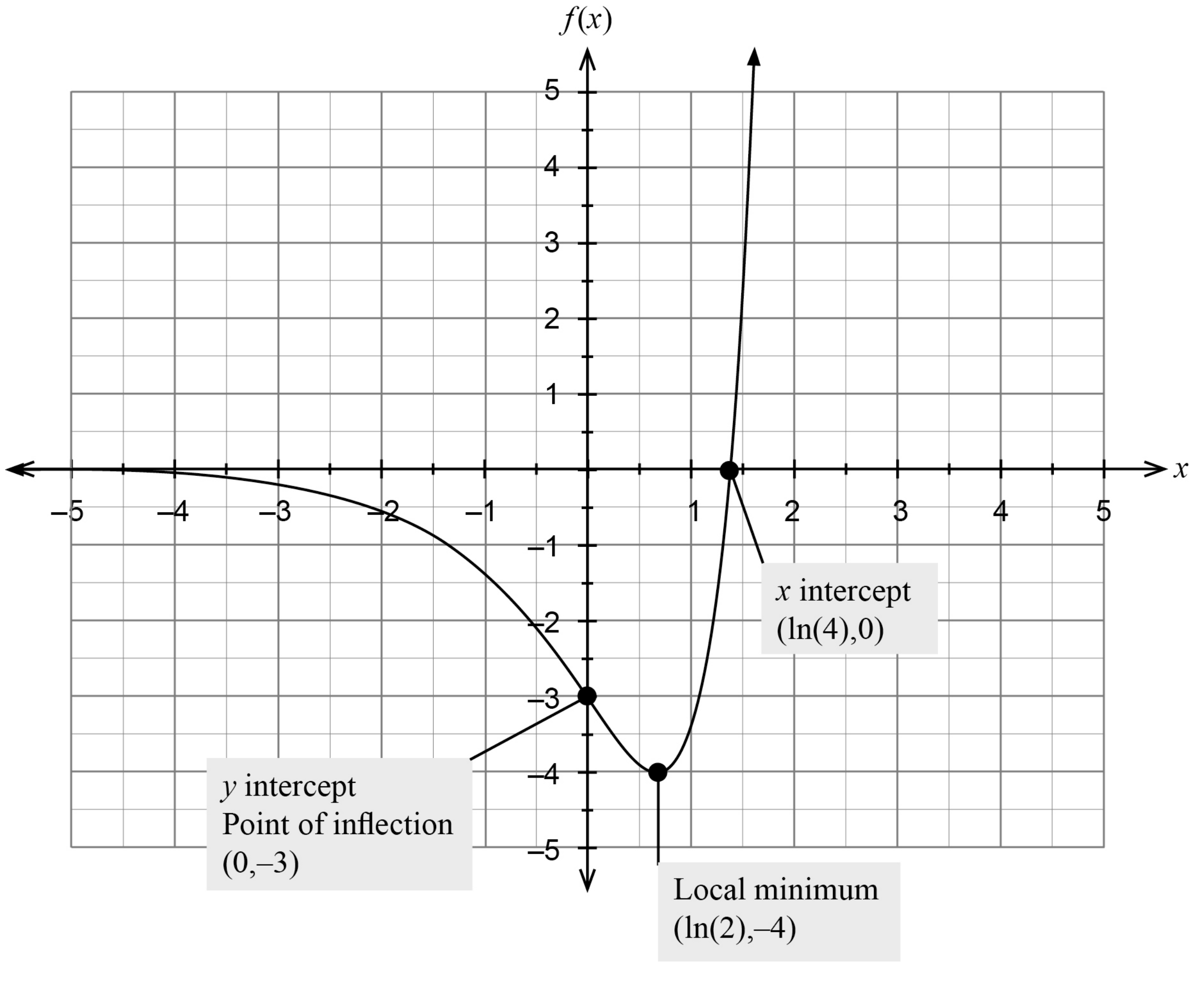
(c) Determine the coordinates of the point(s) of inflection of f. (3 marks)

Solution
$f''(x) = 4e^{2x} - 4e^{x}$ Solve $f''(x) = 0$ $0 = 4e^{2x} - 4e^{x}$ $= 4e^{x}(e^{x} - 1)$ $e^{x} = 1$ $x = \ln(1) = 0$ Substitute $x = 0$ into $f(x)$ $f(\ln(2)) = e^{2(0)} - 4e^{0}$ $= 1 - 4$ $= -3$ Inflection point at $(0, -3)$.
Specific behaviours
✓ differentiates $f'(x)$ correctly and equates to 0   ✓ solves for x ✓ determines y-coordinate of inflection point

(d) Sketch the function f on the axes below, labelling clearly all intercepts, the turning point and point(s) of inflection. Some approximate values of the natural logarithmic function provided in the table below may be helpful. (4 marks)

x	1	2	3	4
$\ln(x)$	0	0.7	1.1	1.4

Solution



Specific behaviours

✓ intercepts correct and labelled
✓ turning point and inflection point correct and labelled
✓ concavity correct
✓ limiting behaviour correct

An *Acknowledgements variation* document is available on the Authority website.

Published by the School Curriculum and Standards Authority of Western Australia
303 Sevenoaks Street
CANNINGTON WA 6107